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Mahdi Boukrouche, Domingo A. Tarzia. Convergence of optimal control problems governed by second kind parabolic variational inequalities. J Control Theory Appl, 2013, 11 (3), pp.422-427. 10.1007/s11768-013-2155-2 . hal-00863292

**HAL Id: hal-00863292**

**<https://hal.science/hal-00863292>**

Submitted on 19 Sep 2013

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# Convergence of optimal control problems governed by second kind parabolic variational inequalities

Mahdi BOUKROUCHE<sup>1</sup>, Domingo A. TARZIA<sup>2</sup>,

<sup>1</sup>Lyon University, UJM F-42023, CNRS UMR 5208, Institut Camille Jordan, 23 Paul Michelon, 42023 Saint-Etienne Cedex 2, France. E-mail: Mahdi.Boukrouche@univ-st-etienne.fr;

<sup>2</sup>Departamento de Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina. E-mail: DTarzia@austral.edu.ar

**Abstract:** We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type. This family is governed by parabolic variational inequalities of the second kind. We prove the strong convergence of the optimal controls and state systems associated to this family to a similar optimal control problem. This work solves the open problem left by the authors in IFIP TC7 CSMO2011.

**Keywords:** Parabolic variational inequalities of the second kind, Aubin compactness arguments, Control border, Convergence of optimal control problems, Tresca boundary conditions, free boundary problems.

## 1 Introduction

The motivation of this paper is to prove the strong convergence of the optimal controls (borders) and state systems associated to a family of second kind parabolic variational inequalities. With this paper, we solve the open question, left in [11] and we generalize our work [10], to study the *Control border*.

To illustrate the problem considered, we consider in the following, just as examples, two free boundary problems which leads to second kind parabolic variational inequalities.

We assume that the boundary of a multidimensional regular domain  $\Omega$  is given by  $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$  with  $meas(\Gamma_1) > 0$  and  $meas(\Gamma_3) > 0$ . We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type whose state system is governed by a free boundary problem with Tresca conditions on a portion  $\Gamma_2$  of the boundary, with a flux  $f$  on  $\Gamma_3$  as the control variable, given by:

### Problem 1.1

$$\begin{aligned} \dot{u} - \Delta u &= g \quad \text{in } \Omega \times (0, T), \\ \left| \frac{\partial u}{\partial n} \right| &< q \Rightarrow u = 0, \quad \text{on } \Gamma_2 \times (0, T), \\ \left| \frac{\partial u}{\partial n} \right| &= q \Rightarrow \exists k > 0 : \quad u = -k \frac{\partial u}{\partial n}, \quad \text{on } \Gamma_2 \times (0, T), \\ u &= b \quad \text{on } \Gamma_1 \times (0, T), \\ -\frac{\partial u}{\partial n} &= f \quad \text{on } \Gamma_3 \times (0, T), \end{aligned}$$

with the initial condition

$$u(0) = u_b \quad \text{on } \Omega,$$

and the compatibility condition on  $\Gamma_1 \times (0, T)$

$$u_b = b \quad \text{on } \Gamma_1 \times (0, T)$$

where  $q > 0$  is the Tresca friction coefficient on  $\Gamma_2$  ([1], [2], [3]). We define the spaces  $\mathcal{F} = L^2((0, T) \times \Gamma_3)$ ,  $V = H^1(\Omega)$ ,  $V_0 = \{v \in V : v|_{\Gamma_1} = 0\}$ ,  $H = L^2(\Omega)$ ,  $\mathcal{H} = L^2(0, T; H)$ ,  $\mathcal{V} = L^2(0, T; V)$  and the closed convex

set  $K_b = \{v \in V : v|_{\Gamma_1} = b\}$ . Let given

$$\begin{aligned} g &\in \mathcal{H}, \quad b \in L^2(0, T, H^{1/2}(\Gamma_1)), \quad f \in \mathcal{F} \\ q &\in L^2((0, T) \times \Gamma_2), \quad q > 0, \quad u_b \in K_b. \end{aligned} \quad (1.1)$$

The variational formulation of *Problem 1.1* leads to the following parabolic variational problem:

**Problem 1.2** Let given  $g, b, q, u_b$  and  $f$  as in (1.1). Find  $u = u_f \in \mathcal{C}(0, T, H) \cap L^2(0, T; K_b)$  with  $\dot{u} \in \mathcal{H}$ , such that  $u(0) = u_b$ , and for  $t \in (0, T)$

$$\begin{aligned} < \dot{u}, v - u > + a(u, u - v) + \Phi(v) - \Phi(u) \geq (g, v - u) \\ &\quad - \int_{\Gamma_3} f(v - u) ds, \quad \forall v \in K_b. \end{aligned}$$

where  $(\cdot, \cdot)$  is the scalar product in  $H$ ,  $a$  and  $\Phi$  are defined by

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx, \quad \text{and } \Phi(v) = \int_{\Gamma_2} q|v| ds. \quad (1.2)$$

The functional  $\Phi$  comes from the Tresca condition on  $\Gamma_2$  [1], [2]. We consider also the following problem where we change, in *Problem 1.1*, only the Dirichlet condition on  $\Gamma_1 \times (0, T)$  by the Newton law or a Robin boundary condition i.e.

### Problem 1.3

$$\begin{aligned} \dot{u} - \Delta u &= g \quad \text{in } \Omega \times (0, T), \\ \left| \frac{\partial u}{\partial n} \right| &< q \Rightarrow u = 0, \quad \text{on } \Gamma_2 \times (0, T), \\ \left| \frac{\partial u}{\partial n} \right| &= q \Rightarrow \exists k > 0 : \quad u = -k \frac{\partial u}{\partial n}, \quad \text{on } \Gamma_2 \times (0, T), \\ -\frac{\partial u}{\partial n} &= h(u - b) \quad \text{on } \Gamma_1 \times (0, T), \\ -\frac{\partial u}{\partial n} &= f \quad \text{on } \Gamma_3 \times (0, T), \end{aligned}$$

with the initial condition

$$u(0) = u_b \quad \text{on } \Omega,$$

and the condition of compatibility on  $\Gamma_1 \times (0, T)$

$$u_b = b \quad \text{on } \Gamma_1 \times (0, T).$$

The variational formulation of the problem (1.3) leads to the following parabolic variational problem

**Problem 1.4** Let given  $g, b, q, u_b$  and  $f$  as in (1.1). For all  $h > 0$ , find  $u = u_{hf}$  in  $\mathcal{C}(0, T, H) \cap \mathcal{V}$  with  $\dot{u}$  in  $\mathcal{H}$ , such that  $u(0) = u_b$ , and for  $t \in (0, T)$

$$\langle \dot{u}, v - u \rangle + a_h(u, u - v) + \Phi(v) - \Phi(u) \geq (g, v - u) - \int_{\Gamma_3} f(v - u)ds + h \int_{\Gamma_1} b(v - u)ds, \quad \forall v \in V,$$

where  $a_h$  is defined by

$$a_h(u, v) = a(u, v) + h \int_{\Gamma_1} uvds.$$

Moreover from [4~7] we have that:  $\exists \lambda_1 > 0$  such that

$$\lambda_h \|v\|_V^2 \leq a_h(v, v) \quad \forall v \in V, \text{ with } \lambda_h = \lambda_1 \min\{1, h\}$$

that is,  $a_h$  is also a bilinear, continuous, symmetric and coercive form  $V \times V$  to  $\mathbb{R}$ . The existence and uniqueness of the solution to each of the above *Problem 1.2* and *Problem 1.4*, is well known see for example [8], [9], [3].

The main goal of this paper is to prove in Section 2 the existence and uniqueness of a family of *optimal control problems* 2.1 and 2.2 where the control variable is given by a boundary condition of Neumann type whose state system is governed by a free boundary problem with Tresca conditions on a portion  $\Gamma_2$  of the boundary, with a flux  $f$  on  $\Gamma_3$  as the control variable, using a regularization method to overcome the nondifferentiability of the functional  $\Phi$ . Then in Section 3 we study the convergence when  $h \rightarrow +\infty$  of the state systems and optimal controls associated to the problem 2.2 to the corresponding state system and optimal control associated to problem 2.1. In order to obtain this last result we obtain an auxiliary strong convergence by using the Aubin compactness arguments see Lemma 3.2. This paper completes our previous paper [10] and solves the open problem left in [11].

Remark here that our study still valid with the bilinear form  $a$  in more general cases, provided that  $a$  must be symmetric, coercive and continuous from  $V \times V$  to  $\mathbb{R}$ .

## 2 Boundary optimal control problems

Let  $M > 0$  be a constant and we define the space

$$\mathcal{F}_- = \{f \in \mathcal{F} : f \leq 0\}.$$

We consider the following Neumann boundary optimal control problems defined by [12~15]

**Problem 2.1** Find the optimal control  $f_{op} \in \mathcal{F}_-$  such that

$$J(f_{op}) = \min_{f \in \mathcal{F}_-} J(f) \quad (2.1)$$

where the cost functional  $J : \mathcal{F}_- \rightarrow \mathbb{R}^+$  is given by

$$J(f) = \frac{1}{2} \|u_f\|_{\mathcal{H}}^2 + \frac{M}{2} \|f\|_{\mathcal{F}}^2 \quad (M > 0) \quad (2.2)$$

and  $u_f$  is the unique solution of the *Problem 1.2* for a given  $f \in \mathcal{F}_-$ .

**Problem 2.2** Find the optimal control  $f_{op_h} \in \mathcal{F}_-$  such that

$$J(f_{op_h}) = \min_{f \in \mathcal{F}_-} J_h(f) \quad (2.3)$$

where the cost functional  $J_h : \mathcal{F}_- \rightarrow \mathbb{R}^+$  is given by

$$J_h(f) = \frac{1}{2} \|u_{hf}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f\|_{\mathcal{F}}^2 \quad (M > 0, h > 0) \quad (2.4)$$

and  $u_{hf}$  is the unique solution of *Problem 1.4* for a given  $f \in \mathcal{F}_-$  and  $h > 0$ .

**Theorem 2.1** Under the assumptions  $g \geq 0$  in  $\Omega \times (0, T)$ ,  $b \geq 0$  on  $\Gamma_1 \times (0, T)$  and  $u_b \geq 0$  in  $\Omega$ , we have the following properties:

- (a) The cost functional  $J$  is strictly convex on  $\mathcal{F}_-$ ,
- (b) There exists a unique optimal control  $f_{op} \in \mathcal{F}_-$  solution of the Neumann boundary optimal control *Problem 2.1*.

**Proof** We give some sketch of the proof, following [10] we generalize for parabolic variational inequalities of the second kind, given in *Problem 1.2*, the estimates obtained for convex combination between  $u_4(\mu) = u_{\mu f_1 + (1-\mu)f_2}$ , and  $u_3(\mu) = \mu u_{f_1} + (1-\mu)u_{f_2}$ , for any two element  $f_1$  and  $f_2$  in  $\mathcal{F}$ . The main difficulty, to prove this result comes from the fact that the functional  $\Phi$  is not differentiable. To overcome this difficulty, we use the regularization method and consider for  $\varepsilon > 0$  the following approach of  $\Phi$  defined by

$$\Phi_\varepsilon(v) = \int_{\Gamma_2} q \sqrt{\varepsilon^2 + |v|^2} ds, \quad \forall v \in V, \quad (2.5)$$

which is Gateaux differentiable, with

$$\langle \Phi'_\varepsilon(w), v \rangle = \int_{\Gamma_2} \frac{q w v}{\sqrt{\varepsilon^2 + |w|^2}} ds \quad \forall (w, v) \in V^2.$$

We define  $u^\varepsilon$  as the unique solution of the corresponding parabolic variational inequality for all  $\varepsilon > 0$ . We obtain that for all  $\mu \in [0, 1]$  we have  $u_4^\varepsilon(\mu) \leq u_3^\varepsilon(\mu)$  for all  $\varepsilon > 0$ .

When  $\varepsilon \rightarrow 0$  we have that: for  $i = 1, \dots, 4$ ,

$$u_i^\varepsilon \rightarrow u_i \text{ strongly in } \mathcal{V} \cap L^\infty(0, T; H). \quad (2.6)$$

As  $f \in \mathcal{F}_-$ ,  $g \geq 0$  in  $\Omega \times (0, T)$ ,  $b \geq 0$  in  $\Gamma_1 \times (0, T)$  and  $u_b \geq 0$  in  $\Omega$ , we obtain by the weak maximum principle that for all  $\mu \in [0, 1]$  we have  $0 \leq u_4(\mu)$ , so following [10] we get

$$0 \leq u_4(\mu) \leq u_3(\mu) \quad \text{in } \Omega \times [0, T], \quad \forall \mu \in [0, 1]. \quad (2.7)$$

Then for all  $\mu \in ]0, 1[$ , and for all  $f_1, f_2$  in  $\mathcal{F}_-$ , and by using  $f_3(\mu) = \mu f_1 + (1-\mu)f_2$  we obtain that:

$$\begin{aligned} & \mu J(f_1) + (1-\mu)J(f_2) - J(f_3(\mu)) = \\ & \frac{1}{2} (\|u_3(\mu)\|_{\mathcal{H}}^2 - \|u_4(\mu)\|_{\mathcal{H}}^2) + \frac{1}{2} \mu(1-\mu) \|u_{f_1} - u_{f_2}\|_{\mathcal{H}}^2 \\ & + \frac{M}{2} \mu(1-\mu) \|f_1 - f_2\|_{\mathcal{F}}^2. \end{aligned} \quad (2.8)$$

Then  $J$  is strictly convex functional on  $\mathcal{F}_-$  and therefore there exists a unique optimal  $f_{op} \in \mathcal{F}_-$  solution of the Neumann boundary optimal control *Problem 2.1*.  $\square$

**Theorem 2.2** Under the assumptions  $g \geq 0$  in  $\Omega \times (0, T)$ ,  $b \geq 0$  in  $\Gamma_1 \times (0, T)$  and  $u_b \geq 0$  in  $\Omega$ , we have the following properties:

- (a) The cost functional  $J_h$  are strictly convex on  $\mathcal{F}_-$ , for all  $h > 0$ ,
- (b) There exists a unique optimal control  $f_{h_{op}} \in \mathcal{F}_-$  solution of the Neumann boundary optimal control *Problem 2.2*, for all  $h > 0$ .

**Proof** We follow a similar method to the one developed in Theorem 2.1 for all  $h > 0$ .  $\square$

### 3 Convergence when $h \rightarrow +\infty$

In this section we study the convergence of the Neumann optimal control *Problem 2.2* to the optimal control *Problem 2.1* when  $h \rightarrow \infty$ . For a given  $f \in \mathcal{F}$  we have first the following result which generalizes [6, 7, 10, 16].

**Lemma 3.1** Let  $u_{hf}$  be the unique solution of the problem 1.4 and  $u_f$  the unique solution of the problem 1.2, then

$$u_{hf} \rightarrow u_f \text{ in } \mathcal{V} \text{ strongly as } h \rightarrow +\infty, \quad \forall f \in \mathcal{F}.$$

**Proof** Following [10], we take  $v = u_f(t)$  in the variational inequality of the problem 1.4 where  $u = u_{hf}$ , and recalling that  $u_f(t) = b$  on  $\Gamma_1 \times ]0, T[$ , taking  $\phi_h(t) = u_{hf}(t) - u_f(t)$  we obtain for  $h > 1$ , that  $\|u_{hf}\|_{\mathcal{V}}$  is also bounded for all  $h > 1$  and for all  $f \in \mathcal{F}$ . Then there exists  $\eta \in \mathcal{V}$  such that (when  $h \rightarrow +\infty$ )

$$u_{hf} \rightharpoonup \eta \text{ weakly in } \mathcal{V}$$

and

$$u_{hf} \rightarrow b \text{ strongly on } L^2((0, T) \times \Gamma_1)$$

so  $\eta(0) = u_b$ .

Let  $\varphi$  be in  $L^2(0, T, H_0^1(\Omega))$  and taking in the variational inequality of the problem 1.4 where  $u = u_{hf}$ ,  $v = u_{hf}(t) \pm \varphi(t)$ , we obtain as  $\|u_{hf}\|_{\mathcal{V}}$  is bounded for all  $h > 1$ , we deduce that  $\|\dot{u}_{hf}\|_{L^2(0, T, H^{-1}(\Omega))}$  is also bounded for all  $h > 1$ . Then we conclude that

$$\left. \begin{aligned} u_{hf} &\rightharpoonup \eta \text{ in } \mathcal{V} \text{ weak, and in } L^\infty(0, T, H) \text{ weak star,} \\ \text{and } \dot{u}_{hf} &\rightharpoonup \dot{\eta} \text{ in } L^2(0, T, H^{-1}(\Omega)) \text{ weak.} \end{aligned} \right\} (3.1)$$

From the variational inequality of the problem 1.4 and taking  $v \in K$  so  $v = b$  on  $\Gamma_1$ , we obtain a.e.  $t \in ]0, T[$

$$\begin{aligned} \langle \dot{u}_{hf}, v - u_{hf} \rangle + a(u_{hf}, v - u_{hf}) - h \int_{\Gamma_1} |u_{hf} - b|^2 ds &\geq \\ \Phi(u_{hf}) - \Phi(v) + (g, v - u_{hf}) - \int_{\Gamma_3} f(v - u_{hf}) ds, \end{aligned}$$

for all  $v \in K$ , then as  $h > 0$  we have a.e.  $t \in ]0, T[$ .

$$\begin{aligned} \langle \dot{u}_{hf}, v - u_{hf} \rangle + a(u_{hf}, v - u_{hf}) &\geq \Phi(u_{hf}) - \Phi(v) + \\ (g, v - u_{hf}) - \int_{\Gamma_3} f(v - u_{hf}) ds, \quad \forall v \in K. \end{aligned} \quad (3.2)$$

So using (3.1) and passing to the limit when  $h \rightarrow +\infty$  we obtain

$$\begin{aligned} \langle \dot{\eta}, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) &\geq (g, v - \eta) \\ - \int_{\Gamma_3} f(v - \eta) ds, \quad \forall v \in K \quad \text{a.e. } t \in ]0, T[, \end{aligned}$$

and  $\eta(0) = u_b$ . Using the uniqueness of the solution of Problem 1.2 we get that  $\eta = u_f$ .

To prove the strong convergence, we take  $v = u_f(t)$  in the variational inequality of the problem 1.4

$$\begin{aligned} \langle \dot{u}_{hf}, u_f - u_{hf} \rangle + a_h(u_{hf}, u_f - u_{hf}) + \Phi(u_f) \\ - \Phi(u_{hf}) &\geq (g, u_f - u_{hf}) + h \int_{\Gamma_1} b(u_f - u_{hf}) ds \\ - \int_{\Gamma_3} f(u_f - u_{hf}) ds, \end{aligned}$$

a.e.  $t \in ]0, T[$ , thus as  $u_f = u_b$  on  $\Gamma_1 \times ]0, T[$ , we put

$\phi_h = u_{hf} - u_f$ , so a.e.  $t \in ]0, T[$  we have

$$\begin{aligned} \langle \dot{\phi}_h, \phi_h \rangle + a(\phi_h, \phi_h) + h \int_{\Gamma_1} |\phi_h|^2 ds + \Phi(u_{hf}) - \Phi(u_f) \\ \leq \langle \dot{u}_f, \phi_h \rangle + a(u_f, \phi_h) + (g, \phi_h) - \int_{\Gamma_3} f \phi_h ds, \end{aligned}$$

so

$$\begin{aligned} \frac{1}{2} \|\phi_h\|_{L^\infty(0, T, H)}^2 + \lambda_h \|\phi_h\|_{\mathcal{V}}^2 + \Phi(u_{hf}) - \Phi(u_f) \\ \leq - \int_0^T \langle \dot{u}_f(t), \phi_h(t) \rangle dt - \int_0^T a(u_f(t), \phi_h(t)) dt \\ + \int_0^T (g(t), \phi_h(t)) dt - \int_0^T \int_{\Gamma_3} f \phi_h ds dt. \end{aligned}$$

Using the weak semi-continuity of  $\Phi$  and the weak convergence (3.1) the right side of the just above inequality tends to zero when  $h \rightarrow +\infty$ , then we deduce the strong convergence of  $\phi_h = u_{hf} - u_f$  to 0 in  $\mathcal{V} \cap L^\infty(0, T, H)$ , for all  $f \in \mathcal{F}_-$  and the proof holds.  $\square$

We prove now the following lemma by using the Aubin compactness arguments. This Lemma 3.2 is very important and necessary which allow us to conclude this paper. Indeed this result is needed to pass to the limit exactly in the last term of the inequality (3.12) in the proof of the main Theorem 3.3.

**Lemma 3.2** Let  $u_{hf_{op_h}}$  the state system defined by the unique solution of Problem 1.4, where the flux  $f$  is replaced by  $f_{op_h}$ . Then, for  $h \rightarrow +\infty$ , we have

$$u_{hf_{op_h}} \rightarrow u_f \text{ in } L^2((0, T) \times \partial\Omega), \quad (3.3)$$

where  $u_f$  is the the state system defined by the unique solution of Problem 1.2 with the flux  $f$  on  $\Gamma_3$ .

**Proof** Let consider the variational inequality of Problem 1.4 with  $u = u_{hf_{op_h}}$  and  $f = f_{op_h}$  i.e.

$$\begin{aligned} \langle \dot{u}_{hf_{op_h}}, v - u_{hf_{op_h}} \rangle + a_h(u_{hf_{op_h}}, v - u_{hf_{op_h}}) + \Phi(v) \\ - \Phi(u_{hf_{op_h}}) &\geq (g, v - u_{hf_{op_h}}) - \int_{\Gamma_3} f_{op_h}(v - u_{hf_{op_h}}) ds \\ + h \int_{\Gamma_1} b(v - u_{hf_{op_h}}) ds, \quad \forall v \in V, \end{aligned} \quad (3.4)$$

and let  $\varphi \in L^2(0, T; H_0^1(\Omega))$ , and set  $v = u_{hf_{op_h}}(t) \pm \varphi(t)$  in (3.4), we get

$$\langle \dot{u}_{hf_{op_h}}, \varphi \rangle = (g, \varphi) - a(u_{hf_{op_h}}, \varphi).$$

By integration in times for  $t \in (0, T)$ , we get

$$\int_0^T \langle \dot{u}_{hf_{op_h}}, \varphi \rangle dt = \int_0^T (g, \varphi) dt - \int_0^T a(u_{hf_{op_h}}, \varphi) dt$$

thus for  $A = (c\|g\|_{\mathcal{H}} + \|u_{hf_{op_h}}\|_{\mathcal{V}})$ , we get

$$\left| \int_0^T \langle \dot{u}_{hf_{op_h}}, \varphi \rangle dt \right| \leq A \|\varphi\|_{L^2(0, T; H_0^1(\Omega))}$$

where  $c$  comes from the Poincaré inequality, and as in Lemma 3.1 we can obtain that  $u_{hf_{op_h}}$  is bounded in  $\mathcal{V}$ , so there exists a positive constant  $C$  such that

$$\|\dot{u}_{hf_{op_h}}\|_{L^2(0, T; H^{-1}(\Omega))} \leq C. \quad (3.5)$$

Using now the Aubin compactness arguments, see for example [17] with the three Banach spaces  $V$ ,  $H^{\frac{2}{3}}(\Omega)$  and  $H^{-1}(\Omega)$ , then

$$u_{hf_{op_h}} \rightarrow u_f \text{ in } L^2(0, T; H^{\frac{2}{3}}(\Omega)).$$

As the trace operator  $\gamma_0$  is continuous from  $H^{\frac{2}{3}}(\Omega)$  to  $L^2(\partial\Omega)$ , then the result follows.  $\square$

We give now, without need to use the notion of adjoint states [14, 18], the convergence result which generalizes the result obtained in [19] for a parabolic variational equalities (see also [18, 20~23]). Other optimal control problems governed by variational inequalities are given in [24~26].

**Theorem 3.3** Let  $u_{hf_{op_h}} \in \mathcal{V}$ ,  $f_{op_h} \in \mathcal{F}_-$  and  $u_{f_{op}} \in \mathcal{V}$ ,  $f_{op} \in \mathcal{F}_-$  be respectively the state systems and the optimal controls defined in the problems (1.4) and (1.2). Then

$$\begin{aligned} & \lim_{h \rightarrow +\infty} \|u_{hf_{op_h}} - u_{f_{op}}\|_{\mathcal{V}} = \\ & = \lim_{h \rightarrow +\infty} \|u_{hf_{op_h}} - u_{f_{op}}\|_{L^\infty(0,T,H)}, \\ & = \lim_{h \rightarrow +\infty} \|u_{hf_{op_h}} - u_{f_{op}}\|_{L^2((0,T) \times \Gamma_1)} = 0, \end{aligned} \quad (3.6)$$

$$\lim_{h \rightarrow +\infty} \|f_{op_h} - f_{op}\|_{\mathcal{F}} = 0. \quad (3.7)$$

**Proof** We have first

$$\begin{aligned} J_h(f_{op_h}) &= \frac{1}{2} \|u_{hf_{op_h}}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f_{op_h}\|_{\mathcal{F}}^2 \leq \\ & \leq \frac{1}{2} \|u_{hf}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f\|_{\mathcal{F}}^2, \end{aligned}$$

for all  $f \in \mathcal{F}_-$ , then for  $f = 0 \in \mathcal{F}_-$  we obtain that

$$J_h(f_{op_h}) = \frac{1}{2} \|u_{hf_{op_h}}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f_{op_h}\|_{\mathcal{F}}^2 \leq \frac{1}{2} \|u_{h0}\|_{\mathcal{H}}^2 \quad (3.8)$$

where  $u_{h0} \in \mathcal{V}$  is the solution of the following parabolic variational inequality

$$\begin{aligned} & \langle \dot{u}_{h0}, v - u_{h0} \rangle + a_h(u_{h0}, v - u_{h0}) + \Phi(v) - \Phi(u_{h0}) \\ & \geq \int_{\Omega} g(v - u_{h0}) dx + h \int_{\Gamma_1} b(v - u_{h0}) ds, \quad a.e. t \in ]0, T[ \end{aligned}$$

for all  $v \in V$  and  $u_{h0}(0) = u_b$ .

Taking  $v = u_b \in K_b$  we get that  $\|u_{h0} - u_b\|_{\mathcal{V}}$  is bounded independently of  $h$ , then  $\|u_{h0}\|_{\mathcal{H}}$  is bounded independently of  $h$ . So we deduce with (3.8) that  $\|u_{hf_{op_h}}\|_{\mathcal{H}}$  and  $\|f_{op_h}\|_{\mathcal{F}}$  are also bounded independently of  $h$ . So there exist  $\tilde{f} \in \mathcal{F}_-$  and  $\eta$  in  $\mathcal{H}$  such that

$$f_{op_h} \rightharpoonup \tilde{f} \text{ in } \mathcal{F}_- \text{ and } u_{hf_{op_h}} \rightharpoonup \eta \text{ in } \mathcal{H} \text{ (weakly)}. \quad (3.9)$$

Taking now  $v = u_{f_{op}}(t) \in K_b$  in Problem (1.4), for  $t \in ]0, T[$ , with  $u = u_{hf_{op_h}}$  and  $f = f_{op_h}$ , we obtain

$$\begin{aligned} & \langle \dot{u}_{hf_{op_h}}, u_{f_{op}} - u_{hf_{op_h}} \rangle + a_1(u_{hf_{op_h}}, u_{f_{op}} - u_{hf_{op_h}}) \\ & + (h-1) \int_{\Gamma_1} u_{hf_{op_h}} (u_{f_{op}} - u_{hf_{op_h}}) ds + \Phi(u_{f_{op}}) \end{aligned}$$

$$\begin{aligned} -\Phi(u_{hf_{op_h}}) & \geq (g, u_{f_{op}} - u_{hf_{op_h}}) + h \int_{\Gamma_1} b(u_{f_{op}} - u_{hf_{op_h}}) ds \\ & - \int_{\Gamma_3} f_{op_h} (u_{f_{op}} - u_{hf_{op_h}}) ds, \quad a.e. t \in ]0, T[. \end{aligned}$$

As  $u_{f_{op}} = b$  on  $\Gamma_1 \times [0, T]$ , taking  $\phi_h = u_{f_{op}} - u_{hf_{op_h}}$  we obtain

$$\begin{aligned} & \frac{1}{2} \|\phi_h\|_{L^\infty(0,T;H)}^2 + \lambda_1 \|\phi_h\|_{\mathcal{V}}^2 + (h-1) \int_0^T \int_{\Gamma_1} |\phi_h(t)|^2 ds dt \\ & \leq \int_0^T \int_{\Gamma_3} f_{op_h} \phi_h ds dt - \int_0^T (g(t), \phi_h(t)) dt \\ & + \int_0^T \int_{\Gamma_2} q |\phi_h(t)| ds dt + \int_0^T \langle \dot{u}_{f_{op}}(t), \phi_h(t) \rangle dt \end{aligned}$$

$$+ \int_0^T a(u_{f_{op}}(t), \phi_h(t)) dt.$$

As  $f_{op_h}$  is bounded in  $\mathcal{F}_-$ , from (3.5)  $\dot{u}_{f_{op}}$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$ , and  $u_{hf_{op_h}}$  is also bounded in  $\mathcal{V}$ , all independently on  $h$ , so there exists a positive constant  $C$  which does not depend on  $h$  such that

$$\begin{aligned} \|\phi_h\|_{\mathcal{V}} &= \|u_{hf_{op_h}} - u_{f_{op}}\|_{\mathcal{V}} \leq C, \quad \|\phi_h\|_{L^\infty(0,T,H)} \leq C \\ & \text{and } (h-1) \int_0^T \int_{\Gamma_1} |u_{hf_{op_h}} - b|^2 ds dt \leq C, \end{aligned}$$

then  $\eta \in \mathcal{V}$  and

$$u_{hf_{op_h}} \rightharpoonup \eta \text{ in } \mathcal{V} \text{ and in } L^\infty(0, T, H) \text{ weak star} \quad (3.10)$$

$$u_{hf_{op_h}} \rightarrow b \text{ in } L^2((0, T) \times \Gamma_1) \text{ strong}, \quad (3.11)$$

so  $\eta(t) \in K_b$  for all  $t \in [0, T]$ .

Now taking  $v \in K$  in Problem (1.4) where  $u = u_{hf_{op_h}}$  and  $f = f_{op_h}$  so

$$\begin{aligned} & \langle \dot{u}_{hf_{op_h}}, v - u_{hf_{op_h}} \rangle + a_h(u_{hf_{op_h}}, v - u_{hf_{op_h}}) + \Phi(v) \\ & - \Phi(u_{hf_{op_h}}) \geq (f_{op_h}, v - u_{hf_{op_h}}) + h \int_{\Gamma_1} b(v - u_{hf_{op_h}}) ds \\ & - \int_{\Gamma_3} f_{op_h} (v - u_{hf_{op_h}}) ds, \quad a.e. t \in ]0, T[ \end{aligned}$$

as  $v \in K_b$  so  $v = b$  on  $\Gamma_1$ , thus we have

$$\begin{aligned} & \langle \dot{u}_{hf_{op_h}}, u_{hf_{op_h}} - v \rangle + a(u_{hf_{op_h}}, u_{hf_{op_h}} - v) + \\ & h \int_{\Gamma_1} |u_{hf_{op_h}} - b|^2 ds + \Phi(u_{hf_{op_h}}) - \Phi(v) - (g, v - u_{hf_{op_h}}) \\ & \leq \int_{\Gamma_3} f_{op_h} (v - u_{hf_{op_h}}) ds \quad a.e. t \in ]0, T[. \end{aligned}$$

Thus

$$\begin{aligned} & \langle \dot{u}_{hf_{op_h}}, u_{hf_{op_h}} - v \rangle + a(u_{hf_{op_h}}, u_{hf_{op_h}} - v) \\ & + \Phi(u_{hf_{op_h}}) - \Phi(v) \leq -(g, v - u_{hf_{op_h}}) \\ & - \int_{\Gamma_3} f_{op_h} (v - u_{hf_{op_h}}) ds \quad a.e. t \in ]0, T[. \end{aligned} \quad (3.12)$$

Using Lemma 3.2, (3.9) and (3.10), we deduce that [3, 27]

$$\begin{aligned} & \langle \dot{\eta}, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \geq (f, v - \eta) \\ & - \int_{\Gamma_3} \tilde{f} (v - \eta) ds, \quad \forall v \in K, \quad a.e. t \in ]0, T[, \end{aligned}$$

so also by the uniqueness of the solution of Problem (1.2) we obtain that

$$u_{\tilde{f}} = \eta. \quad (3.13)$$

We prove that  $\tilde{f} = f_{op}$ . Indeed we have

$$\begin{aligned} J(\tilde{f}) &= \frac{1}{2} \|\eta\|_{\mathcal{H}}^2 + \frac{M}{2} \|\tilde{f}\|_{\mathcal{F}}^2 \\ & \leq \liminf_{h \rightarrow +\infty} \left\{ \frac{1}{2} \|u_{hf_{op_h}}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f_{op_h}\|_{\mathcal{F}}^2 \right\} \\ & = \liminf_{h \rightarrow +\infty} J_h(f_{op_h}) \\ & \leq \liminf_{h \rightarrow +\infty} J_h(f) = \liminf_{h \rightarrow +\infty} \left\{ \frac{1}{2} \|u_{hf}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f\|_{\mathcal{F}}^2 \right\} \end{aligned}$$

so using now the strong convergence  $u_{hf} \rightarrow u_f$  as  $h \rightarrow +\infty$ ,  $\forall f \in \mathcal{F}_-$  (see Lemma 3.1), we obtain that

$$J(\tilde{f}) \leq \liminf_{h \rightarrow +\infty} J_h(f_{op_h}) \leq \frac{1}{2} \|u_f\|_{\mathcal{H}}^2 + \frac{M}{2} \|f\|_{\mathcal{F}}^2$$

$$= J(f), \quad \forall f \in \mathcal{F}_- \quad (3.14)$$

then by the uniqueness of the optimal control *Problem* (1.2) we get

$$\tilde{f} = f_{op}. \quad (3.15)$$

Now we prove the strong convergence of  $u_{hf_{op_h}}$  to  $\eta = u_f$  in  $\mathcal{V} \cap L^\infty(0, T; H) \cap L^2(0, T; L^2(\Gamma_1))$ , indeed taking  $v = \eta$  in *Problem* (1.4) where  $u = u_{hf_{op_h}}$  and  $f = f_{op_h}$ , as  $\eta(t) \in K$  for  $t \in [0, T]$ , so  $\eta = b$  on  $\Gamma_1$ , we obtain

$$\begin{aligned} & \frac{1}{2} \|u_{hf_{op_h}} - \eta\|_{L^\infty(0, T; H)}^2 + \lambda_1 \|u_{hf_{op_h}} - \eta\|_{\mathcal{V}}^2 + \\ & \int_0^T \{\Phi(u_{hf_{op_h}}) - \Phi(\eta)\} dt + \tilde{h} \|u_{hf_{op_h}} - \eta\|_{L^2((0, T) \times \Gamma_1)}^2 \\ & \leq \int_0^T (g, u_{hf_{op_h}}(t) - \eta(t)) dt - \int_0^T \langle \dot{\eta}, u_{hf_{op_h}} - \eta \rangle dt + \\ & \int_0^T a(\eta(t), \eta(t) - u_{hf_{op_h}}(t)) - \int_{\Gamma_3} f_{op_h}(u_{hf_{op_h}} - \eta) ds dt. \end{aligned}$$

where  $\tilde{h} = h - 1$

Using (3.10) and the weak semi-continuity of  $\Phi$  we deduce that

$$\begin{aligned} \lim_{h \rightarrow +\infty} \|u_{hf_{op_h}} - \eta\|_{L^\infty(0, T; H)} &= \lim_{h \rightarrow +\infty} \|u_{hf_{op_h}} - \eta\|_{\mathcal{V}} \\ &= \|u_{hf_{op_h}} - \eta\|_{L^2((0, T) \times \Gamma_1)} = 0, \end{aligned}$$

and with (3.13) and (3.15) we deduce (3.6). Then from (3.14) and (3.15) we can write

$$\begin{aligned} J(f_{op}) &= \frac{1}{2} \|u_{f_{op}}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f_{op}\|_{\mathcal{F}}^2 \leq \liminf_{h \rightarrow +\infty} J_h(f_{op_h}) \\ &= \liminf_{h \rightarrow +\infty} \left\{ \frac{1}{2} \|u_{hf_{op_h}}\|_{\mathcal{H}}^2 + \frac{M}{2} \|f_{op_h}\|_{\mathcal{F}}^2 \right\} \\ &\leq \lim_{h \rightarrow +\infty} J_h(f_{op}) = J(f_{op}) \end{aligned} \quad (3.16)$$

and using the strong convergence (3.6), we get

$$\lim_{h \rightarrow +\infty} \|f_{op_h}\|_{\mathcal{F}} = \|f_{op}\|_{\mathcal{F}}. \quad (3.17)$$

Finally as

$$\|f_{op_h} - f_{op}\|_{\mathcal{F}}^2 = \|f_{op_h}\|_{\mathcal{F}}^2 + \|f_{op}\|_{\mathcal{F}}^2 - 2(f_{op_h}, f_{op}) \quad (3.18)$$

and by the first part of (3.9) we have

$$\lim_{h \rightarrow +\infty} (f_{op_h}, f_{op}) = \|f_{op}\|_{\mathcal{F}}^2,$$

so from (3.17) and (3.18) we get (3.7). This ends the proof.  $\square$

**Corollary 3.4** Let  $u_{hf_{op_h}}$  in  $\mathcal{V}$ ,  $f_{op_h}$  in  $\mathcal{F}_-$ ,  $u_{f_{op}}$  in  $\mathcal{V}$  and  $f_{op}$  in  $\mathcal{F}_-$  be respectively the state systems and the optimal controls defined in the problems (1.4) and (1.2). Then

$$\lim_{h \rightarrow +\infty} |J_h(f_{op_h}) - J(f_{op})| = 0.$$

**Proof** It follows from the definitions (2.1) and (2.2), and the convergences (3.6) and (3.7).  $\square$

## 4 Conclusion

The main difference here with our work [10] where the control variable was the function  $g$ , is that we consider here as a control variable the function  $f$  given by the Neumann boundary condition on  $\Gamma_3$ . This change induce in the

variational problems 1.2 and 1.4, and also in the proofs of Lemma 3.1 and Theorem 3.3, a new integral term on  $\Gamma_3$ . The main difficulty here is in Section 3 and the question is exactly how to pass to the limit for  $h \rightarrow +\infty$  in the last integral term on  $\Gamma_3$  in (3.12). To overcome this main difficulty we have introduced the new Lemma 3.2, which is the key of our problem. The idea of Lemma 3.1 and Theorem 3.3 and their proofs are indeed similar to those of our work [10] with the differences and difficulties mentioned just above.

## Acknowledgements

This paper was partially sponsored by the Institut Camille Jordan ST-Etienne University for first author and the project PICTO Austral # 73 from ANPCyT and Grant AFOSR FA9550-10-1-0023 for the second author.

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